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ALFVEN INERTIAL INTERNAL GRAVITY WAVES BY RANK MATRIX METHOD

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ABSTRACT

In this paper we have analyzed the effect of rotation and magnetic field on linear and nonlinear internal gravity waves called Alfven inertial internal gravity waves propagating in an exponentially stratified incompressible and infinitely conducting fluid. The problem is governed by nine nonlinear inhomogeneous PDE's which has been reduced to third order ODE's by using traveling wave solutions and some first integrals. The resulting system can be analysed in Phase-plane by identifying singularities. We have solved the same system by Rank matrix method. The Rank matrix gives us new solutions which are not obtained by Achala [2001].

Keywords: *travelling wave, rotating stratified fluid, inhomogeneous systems, phase function, internal gravity waves, Rank matrix method.*

I. INTRODUCTION

Density stratification and gravity play important role in wave generation in nonhomogeneous fluid. Gravity waves in homogeneous fluid exist only when there is a free surface, which is nothing but surface fluid discontinuity, i. e density stratification. Gravity acts as restoring force if there exist density stratification, which in turn leads to oscillations. For an incompressible fluid to be stable if the density of displaced fluid whose position is lower than old is greater and unstable if it is lesser. Thus oscillation or wave motion is possible only if the stratification is statically stable i.e density decreases with height. For stability of compressible fluid entropy decreases with elevation and wave motion exists only in stably stratified fluid.

Internal waves are waves which occur in the interior of a fluid where gravity is the restoring force. The density differences in the interior of the fluid are tiny compared to those at the surface which are present in the oceans and the atmosphere make it possible to have very large internal waves with large currents hence they can transport material in the ocean or atmosphere along with them. *Inertial waves* are a type of mechanical *wave* possible in rotating fluids commonly seen at the beach or in the bathtub. Inertial waves flow through the interior of the fluid, not at the surface and restoring force for inertial waves is the Coriolis force**.** Most commonly they are observed in atmospheres, oceans, lakes, and laboratory experiments. Rossby waves, geostrophic currents, and geostrophic winds are examples of inertial waves. Inertial waves are also likely to exist in the molten core of the rotating Earth [1]. The linear theory of inertial waves is known well [2, 3] while the influence of nonlinear effects of wave interactions are subject of many recent theoretical and experimental studies.

Waves in electrically conducting fluids occurring as a result of the stability imparted by magnetic fields is known as hydromagnetic or Alfvén waves [4]. They are found in plasmas or fluids with high electrical conductivity, such as the solar corona and Earth's magnetosphere and core. It is therefore a matter of considerable geophysical and astrophysical importance to understand and be able to quantitatively model such waves and their generalizations that occur when both magnetic fields and rotation are present.

The basic mechanism underlying waves in electrically conducting fluids permeated by magnetic fields was elucidated by Alfvén [5]. He described a scenario whereby magnetic tension and inertial effects give rise to oscillations and travelling waves, which are known as Alfvén waves in his honour. Lehnert [6] deduced that rapid rotation of a hydromagnetic system would lead to the splitting of plane Alfvén waves into two circularly polarized, transverse, waves. He realized these would have very different timescales if the frequency of Inertial waves was much larger than that of pure Alfvén waves in the system. Here, such waves will be collectively be referred to as Magnetic Coriolis (MC) waves. Chandrasekhar[7] studied the effects of buoyancy on rotating hydromagnetic systems, though he focused primarily on axisymmetric motions. Braginsky [9,10] described the influence of density stratification and convection driving non-axisymmetric waves naming these Magnetic Archimedes Coriolis (MAC) waves.

A more focused technical reviews of the subject are given by Roberts and Soward [11], Acheson and Hide [12a,12b], Eltayeb [13a,13b], Fearn Roberts and Soward [14], Proctor[15], Zhang and Schubert [16] , Soward and Dormy [17]. Moffatt's monograph [18] is best material for essential reading.

Traveling wave solutions of inhomogeneous systems as quasi-simple waves has been studied extensively by many authors (see Courant and Friedrichs [19], Schindler[20]). The concept of simple integral elements in this context was introduced by Grundland [21] with a view to studying the properties of solutions which depend on the nature of the inhomogeneity. Extensive work in this regard has been carried out by many authors both for linear and nonlinear theory. Linear work is carried out by Bretherton [22], Booker and Bretherton [23], Jones [24], Acheson [25a, 25b, 26a, 26b], Grimshaw [27], Rudraiah and Venkatachalappa [28a, 28b, 28c] and others.

Nonlinear theory was carried out by Seshadri and Sachdev [29] for acoustic gravity waves in compressible isothermal atmosphere, Venkatachalappa, Rudraiah and Sachdev [30], Rudraiah, Sachdev and Venkatachalappa [31] for rotating compressible stratified fluid, Venkatachalappa, Achala and Sachdev [32] for incompressible, rotating, stratified flows as limiting case of [30,31]. Rudraiah and Venkatachalappa (1979) have studied the internal Alfven – inertial gravity waves with basic flow in different from zero subject to infinitesimal perturbations and obtained solutions near critical levels. Later Venkatachalappa, Rudraiah and Sachdev [30] have studied the propagation of linear and nonlinear traveling waves in a compressible rotating atmosphere. Venkatachalappa, Achala and Sachdev [32] have investigated the propagation of linear and nonlinear traveling waves in an exponentially stratified incompressible rotating fluid. Venkatachalappa, Rudraiah and Sachdev [31] have studied the propagation of linear and nonlinear hydro magnetic waves in an exponentially stratified incompressible medium. In this chapter we analyze the effect of rotation and magnetic field on linear and nonlinear internal gravity waves called Alfven inertial internal gravity waves propagating in an exponentially stratified incompressible and infinitely conducting fluid. The waves under study are governed by a system of nine nonlinear inhomogeneous PDE's. We seek travelling wave solutions of this system.

II. BASIC EQUATIONS

The model considered is in Cartesian co-ordinate system with x- and y-axes in the horizontal plane and z-axes along vertical direction. The study is on quasi simple internal gravity waves in an incompressible, infinitely conducting stratified fluid rotating with uniform angular velocity Ω about a vertical axis in the presence of an applied variable magnetic field $H_0(z)$ in the x- direction. The equations governing unsteady system are:

$$\rho\left[\frac{D\vec{q}}{Dt}+2\vec{\Omega}\times\vec{q}\right]+\nabla P-\rho\vec{g}-\mu\left(\vec{H}.\nabla\right)\vec{H}=0 \quad (1)$$

$$\frac{D\rho}{Dt}=0 \quad (2)$$

$$\nabla.\vec{q}=0 \quad (3)$$

$$\frac{D\vec{H}}{Dt}-\left(\vec{H}.\nabla\right)\vec{q}=0 \quad (4)$$

$$\nabla.\vec{H}=0 \quad (5)$$

where $\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$, $P=p+\frac{\mu H^2}{2}$ is the total pressure, p the hydrodynamic pressure, $\vec{q}$ with components (u,v,w) the fluid velocity, ρ the density, $\vec{g}$ the acceleration due to gravity, $\vec{\Omega}$ the angular velocity, $\vec{H}$ the magnetic field with components (H_x,H_y,H_z) in the x,y,z directions respectively and μ is the permeability. We assume that the undisturbed fluid has density $\rho_0(z)$ and an applied magnetic field $H_o(z)$ of the form,

$$\rho_0(z)=\rho_c\exp\left(-z/\vec{H}\right) \quad (6)$$

$$H_0(z)=H_c\exp\left(-z/2\vec{H}\right) \quad (7)$$

where $\vec{H}$ is the scale height, ρ_c and H_c are the reference density and magnetic field at $z=0$. From equation (1) we find that the basic pressure $p_0(z)$ is given by

$$p_o(z)=p_c\exp\left(-z/\vec{H}\right) \quad (8)$$

where $p_c=g\vec{H}\rho_c$ is the hydrodynamic balance. We non dimensionalise the equations (1) - (5) using $H,\left(g/\vec{H}\right)^{\frac{1}{2}},\left(g\vec{H}\right)^{\frac{1}{2}},p_c e^{\left(-z/\vec{H}\right)},\rho_c e^{\left(-z/2\vec{H}\right)}$ as the scales for length, time, velocity, pressure, density and magnetic field respectively. We thus have

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-2\Omega v+\frac{1}{\rho}\frac{\partial P}{\partial x}$$
$$-\frac{A^2}{\rho}\left(H_x\frac{\partial H_x}{\partial x}+H_y\frac{\partial H_x}{\partial y}+H_z\frac{\partial H_x}{\partial z}-\frac{H_xH_z}{2}\right)=0, \quad (9)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+2\Omega u+\frac{1}{\rho}\frac{\partial P}{\partial y}$$
$$-\frac{A^2}{\rho}\left(H_x\frac{\partial H_y}{\partial x}+H_y\frac{\partial H_y}{\partial y}+H_z\frac{\partial H_y}{\partial z}-\frac{H_yH_z}{2}\right)=0, \quad (10)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{1}{\rho}\frac{\partial P}{\partial x}+\left(1-\frac{p}{\rho}\right)$$
$$-\frac{A^2}{\rho}\left(H_x\frac{\partial H_z}{\partial x}+H_y\frac{\partial H_z}{\partial y}+H_z\frac{\partial H_z}{\partial z}-\frac{H_z^2}{2}\right)=0, \quad (11)$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}-\rho w=0, \quad (12)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \quad (13)$$

$$\frac{\partial H_x}{\partial t}+\left(u\frac{\partial H_x}{\partial x}+v\frac{\partial H_x}{\partial y}+w\frac{\partial H_x}{\partial z}\right)=0, \quad (14)$$

$$\frac{\partial H_y}{\partial t}+\left(u\frac{\partial H_y}{\partial x}+v\frac{\partial H_y}{\partial y}+w\frac{\partial H_y}{\partial z}\right)=0, \quad (15)$$

$$\frac{\partial H_z}{\partial t}+\left(u\frac{\partial H_z}{\partial x}+v\frac{\partial H_z}{\partial y}+w\frac{\partial H_z}{\partial z}\right)=0, \quad (16)$$

$$\frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}-\frac{H_z}{2}=0, \quad (17)$$

where $A^2=\frac{\mu H_c^2}{\rho_c gH}$ physically this represents non-dimensional Alfven velocity.

We seek traveling wave solutions of the (9) – (17) in the form

$$u=u(\phi), v=v(\phi), w=w(\phi), \rho=\rho(\phi), P=P(\phi),$$

$$H_x=H_x(\phi), H_y=H_y(\phi), H_z=H_z(\phi)$$

$$\phi=\frac{x}{\lambda_1}+\frac{y}{\lambda_2}+\frac{z}{\lambda_3}-t \quad (18)$$

where $\lambda_1,\lambda_2,\lambda_3$ are arbitrary constants, that can be considered as wave lengths in x, y, z directions, the initial conditions are

$$u=v=w=H_y=H_z=0, p=\rho=H_x=1 \quad (19)$$

Hence equations (9) - (17) now become

$$u_\phi+2\Omega v-\frac{P_\phi}{\rho\lambda_1}+\frac{A^2}{2}\left[B(H_x)_\phi-\frac{H_zH_x}{2}\right]=0, \quad (20)$$

$$v_\phi-2\Omega u-\frac{P_\phi}{\rho\lambda_2}+\frac{A^2}{2}\left[B(H_y)_\phi-\frac{H_zH_y}{2}\right]=0, \quad (21)$$

$$w_\phi-\frac{P_\phi}{\rho\lambda_3}+\frac{p}{\rho}-1+\frac{A^2}{2}\left[B(H_z)_\phi-\frac{H_z^2}{2}\right]=0, \quad (22)$$

$$\rho_\phi-\rho w=0, \quad (23)$$

$$(H_x)_\phi+Bu_\phi+\frac{wH_x}{2}=0, \quad (24)$$

$$(H_y)_\phi+Bv_\phi+\frac{wH_y}{2}=0, \quad (25)$$

$$(H_z)_\phi+Bw_\phi+\frac{wH_z}{2}=0, \quad (26)$$

$$B_\phi - \frac{H_z}{2} = 0 \quad , \tag{27}$$

where $B = \frac{H_x}{\lambda_1} + \frac{H_y}{\lambda_2} + \frac{H_z}{\lambda_3}$,

By suitably combining equations (20) - (27), we reduce this eighth order system to third order system,

$$w_\phi = \frac{-\bar{n}\lambda_3(k-1) - 2\Omega\xi}{n\lambda_3(1 - A^2 Q)}, \tag{28}$$

$$k_\phi = kw - \frac{(1-k)}{n\lambda_3} - \frac{2\Omega\xi}{n}, \tag{29}$$

$$\xi_\phi = \frac{2\Omega w}{\lambda_3(1 - A^2 Q)}, \tag{30}$$

where

$$k = \frac{p}{\rho}, \bar{n} = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}, n = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} \tag{31}$$

the only singular point of the system (28) - (30) in (ξ, w, k) space is ,

$$\xi = 0, k = 1, w = 0 . \tag{32}$$

The solution can analysed near these singular points by linearising the system.

III. METHODOLOGY: RANK MATRIX METHOD

One useful application of calculating the rank of a matrix is the computation of the number of solutions of a system of linear equations. According to the Roche-Capelli theorem, the system is inconsistent if the rank of the augmented matrix is greater than the rank of the coefficient matrix. If, on the other hand, ranks of these two matrices are equal, the system must have at least one solution. The solution is unique if and only if the rank equals the number of variables. Otherwise the general solution has *k* free parameters where *k* is the difference between the number of variables and the rank.

Let us think of a *r* x *c* matrix as a set of *r* row vectors, each having *c* elements; or let it be a set of *c* column vectors, each having *r* elements.

The rank of a matrix is defined as the maximum number of linearly independent *column* vectors in the matrix or the maximum number of linearly independent *row* vectors in the matrix.

For a matrix of order *r* x *c* matrix,

- If *r* is less than *c*, then the maximum rank of the matrix is *r*.
- If *r* is greater than *c*, then the maximum rank of the matrix is *c*.
- The rank of a matrix would be zero only if the matrix had no elements. If a matrix had even one element, its minimum rank would be one.
- To find the rank of a matrix, we simply transform the matrix to its row echelon form and count the number of non-zero rows.

The system (20) to (27) can be written in the following form:

$$A_{ij}\frac{dU_j}{d\phi} = B_i \tag{33}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{\rho\lambda_1} & 0 & \frac{A^2B}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{\rho\lambda_2} & 0 & 0 & \frac{A^2B}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{\rho\lambda_3 0} & 0 & 0 & 0 & \frac{A^2B}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ B & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & B & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\lambda_1} & \frac{1}{\lambda_2} & \frac{1}{\lambda_3} \end{bmatrix} U = \begin{bmatrix} u \\ v \\ w \\ \rho \\ p \\ H_x \\ H_y \\ H_z \end{bmatrix} B = \begin{bmatrix} \frac{A^2}{4}H_xH_z - 2\Omega v \\ \frac{A^2}{4}H_yH_z + 2\Omega u \\ \frac{A^2}{4}H_z^2 - \left(\frac{p}{\rho} - 1\right) \\ \rho w \\ -\frac{wH_x}{2} \\ -\frac{wH_y}{2} \\ -\frac{wH_z}{2} \\ \frac{H_z}{2} \end{bmatrix} \tag{34}$$

This system is then studied both when the coefficient matrix of the left–hand side of the algebraic system is of maximum rank and when it is lowers than that. In the latter case Kronecker - Capelli theorem leads to certain conditions on the unknown functions, which are, in fact, the intermediate integrals. The system does not have any solutions if the rank of A is less than that of $[A, B]$. Reducing the augmented matrix $[A, B]$ to echelon form we obtain,

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$$
\left[\begin{array}{cccccccccc}
1 & 0 & 0 & -\frac{1}{\rho\lambda_1} & 0 & \frac{A^2B}{2} & 0 & 0 & : & \frac{A^2}{4}H_zH_x-2\Omega v \\
0 & 1 & 0 & -\frac{1}{\rho\lambda_2} & 0 & 0 & \frac{A^2B}{2} & 0 & : & \frac{A^2}{4}H_zH_y-2\Omega u \\
0 & 0 & 1 & -\frac{1}{\rho\lambda_3} & 0 & 0 & 0 & \frac{A^2B}{2} & : & \frac{A^2}{4}H_z^2+\left(1-\frac{p}{\rho}\right) \\
0 & 0 & 0 & -\frac{B}{\rho\lambda_1} & 1 & \frac{A^2B^2}{2}-1 & 0 & 0 & : & \frac{A^2}{4}BH_z^2+B\left(1-\frac{p}{\rho}\right)+\frac{wH_z}{2} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & : & \rho w \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\lambda_2}\left(\frac{A^2B^2}{2}-1\right) & \frac{1}{\lambda_3}\left(\frac{A^2B^2}{2}-1\right) & 0 & : & \begin{array}{c}\frac{A^2B}{4\lambda_1}H_z^2+\frac{B}{\lambda_1}\left(1-\frac{p}{\rho}\right)+\frac{wH_z}{2\lambda_1}\\ -\frac{A^2B}{4\lambda_3}H_zH_x+\frac{2\Omega Bv}{\lambda_3}+\frac{wH_x}{2\lambda_3}\end{array} \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\lambda_1\lambda_3}\left(\frac{A^2B^2}{2}-1\right) & -\frac{1}{\lambda_1\lambda_2}\left(\frac{A^2B^2}{2}-1\right) & : & \begin{array}{c}\frac{A^2B}{4\lambda_2}H_z^2+\frac{B}{\lambda_2}\left(1-\frac{p}{\rho}\right)+\frac{wH_z}{2\lambda_2}\\ -\frac{A^2B}{4\lambda_3}H_zH_y-\frac{2\Omega Bu}{\lambda_3}-\frac{wH_y}{2\lambda_3}\end{array} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & n\left(\frac{A^2B^2}{2}-1\right) & : & \begin{array}{c}\bar{n}m+\bar{n}l+\frac{wH_z}{2}\bar{n}-\frac{A^2B}{4\lambda_2\lambda_3}H_zH_y\\ -\frac{2\Omega Bu}{\lambda_2\lambda_3}-\frac{wH_y}{2\lambda_2\lambda_3}+\frac{H_z}{2\lambda_3}\left(\frac{A^2B^2}{2}-1\right)\\ -\frac{A^2B}{4\lambda_1\lambda_3}H_zH_x+\frac{2\Omega Bv}{\lambda_1\lambda_3}+\frac{wH_x}{2\lambda_1\lambda_3}\end{array}
\end{array}\right]
$$

where,

$\bar{n}=\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}$, (36)

$n=\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}+\frac{1}{\lambda_3^2}$, (37)

$m=\frac{A^2B}{4}H_z^2$, (38)

$l=B\left(1-\frac{p}{\rho}\right)$. (39)

We can observe that the conditions for matrix $[A]$ and $[A,B]$ to be of maximum rank equal to 8 are

$\frac{A^2B^2}{2}-1\neq 0\ or\ n\neq 0$. (40)

Discussion of other conditions for full system is not so easy so now, let us discuss for the case $\lambda_3 \to \infty$.

For rank to be 7 the conditions are:

$\frac{A^2B^2}{2} - 1 = 0$ and $m + l + \frac{wH_z}{2} = 0$ or $\bar{n} = 0$, (41)

or

$\frac{A^2B^2}{2} - 1 = 0$ and $(H_z)_\emptyset = 0$, (42)

or

$\frac{A^2B^2}{2} - 1 = 0$ and $(H_z)_\emptyset = 0$, $m + l + \frac{wH_z}{2} = 0$ or $\bar{n} = 0$, (43)

Case i: $\frac{A^2B^2}{2} - 1 \neq 0.$

Using this condition and solving (28) for the derivatives $U_{j,\emptyset}$, we get the solution which exactly matches with that obtained by Achala (Thesis 2001).

For rank to be seven, we have the following cases,

Case ii: $\frac{A^2B^2}{2} - 1 = 0$ and $\bar{n}t + \bar{n}l + \frac{wH_z}{2}\bar{n} = 0$

Solving the system (20) to (27) using the above condition we get

$B = \frac{\sqrt{2}}{A}$; $w = \frac{A}{\sqrt{2}} H_z$. (44)

These are very important new solutions in real plane.

Case iii: $\frac{A^2B^2}{2} - 1 = 0$ and $(H_z)_\emptyset = 0$

Solving the system (4.20) to (4.27) using the above condition we get

$\frac{H_x}{\lambda_1} + \frac{H_y}{\lambda_2} = C_1$, (45)

where C_1 is a constant.

Case iv: $\frac{A^2B^2}{2} - 1 = 0$ and $(H_z)_\emptyset = 0$, $\bar{n}m + \bar{n}l + \frac{wH_z}{2}\bar{n} = 0$

Using the above condition, system (4.20) to (4.27) reduces to

$w = \frac{A}{\sqrt{2}} H_z$. (46)

The solutions obtained in **case ii** and **case iv** are same. These are new solutions and are not reported by any previous researchers. Thus we observe that the rank matrix gives all possible analytic solutions. For a big system this method is easy and efficient for obtaining intermediate integrals.

IV. INFERENCE

In this paper we are dealing with nine nonlinear equations with nine unknowns. We have obtained new analytic solutions to nonlinear internal gravity waves called Alfven inertial internal gravity waves propagating in an exponentially stratified incompressible and infinitely conducting fluid by Rank matrix method. It is a very powerful method which gives all possible existing analytic solutions.

We have obtained new conditional solutions of the problem which are given in equation (44) to (46).

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